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A theoretical analysis of filtration mechanisms has been made for fibrous filters in the region of maximum penetration. The theory is based on a boundary layer approach using the Kuwabara flow field to account for the interference effects of neighboring fibers. An improved expression for the diffusion and interception filtration efficiencies has been derived that compares well with the existing theories. A comparison of the developed theory with experimental data also confirms the validity of the present work.

NOMENCLATURE

A	constant
b	outer cell radius in the cell model (cm or μm)
D	diffusion coefficient (cm^2/sec)
D_f	fiber diameter (cm)
D_p	particle diameter (cm or μm)
K	hydrodynamic factor of Kuwabara flow $= -\frac{1}{2}\ln\alpha - \frac{3}{4} + \alpha - \frac{1}{4}\alpha^2$
K_s	hydrodynamic factor of Spielman and Goren flow (see Table 1)
K_0	hydrodynamic factor of Lamb's flow $= 2.00 - \ln \mathrm{Re}$
k_0, k_1	modified Bessel function
M'	dimensionless rate of diffusion to the fiber surface
M_∞'	maximum value of M'
n	particle concentration (number/cm^3)
n_0	particle concentration at a distance far from the fiber (number/cm^3)
n'	dimensionless particle concentration $= n/n_0$
Pe	Peclet number $= uD_f/D$
R	interception parameter, diameter ratio of particle to fiber, or dimensionless particle radius $= R_p/R_f$
Re	Reynolds number $= \rho_a u D_f/\mu$
R_f	fiber radius (cm or μm)
R_p	particle radius (cm or μm)
r	position coordinate in the radial direction
r'	dimensionless radial coordinate
U_0	face velocity, undisturbed air velocity (cm/sec)
u	flow velocity toward fiber on the boundary and at $\theta = 0$ in the cell model (cm/sec)
u_r	radial component flow velocity (cm/sec)
u_θ	circumferential component flow velocity (cm/sec)
u_r'	dimensionless radial component flow velocity $= u_r/u$

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u_0' dimensionless circumferential component flow velocity $=u_0/u$

Y thickness of streamline in which all the particles are filtered by fiber (cm or μm)

α volume fraction, solidity, or packing density of filter

β_1 coefficient of diffusional filtration efficiency [see Eq. (35)]

β_2 coefficient of interceptional filtration efficiency

$\Gamma(c)$ gamma function of c

η single fiber efficiency

η_D single fiber efficiency due to diffusion

η_R single fiber efficiency due to interception

θ position coordinate in the circumferential direction

λ function of α (see Table 1)

μ air viscosity (poise)

ρ boundary layer radius (cm or μm)

ρ_a air density (g/cm^3)

ρ' dimensionless boundary layer radius $=\rho/R_f$

ψ stream function

ψ_ρ stream function passing through the outer diffusion boundary layer

ψ' dimensionless stream function $=\psi/uR_f$

$\Delta\psi'$ error involved in approximation for dimensionless stream function

INTRODUCTION

There have been extensive theoretical investigations of filtration mechanisms since the times of Langmuir (1942). One of the difficulties that remained in the filtration theories until 1963 concerned the treatment of the neighboring fiber interference effect. A second theoretical difficulty has been related to the treatment of the nonideal effects in fiber orientation. The random orientation of fibers and the inhomogeneity of the fibers cause the flow for real filters to deviate from the ideal flow patterns assumed theoretically.

The neighboring fiber interference effect was adequately taken into account first by Fuchs and Stechkina (1963) using the Kuwabara–Happel flow field (Happel, 1959; Kuwabara, 1959). The Kuwabara–Happel flow field is based on the solution of the Navier–Stokes equation for the case of viscous flow around a cylinder by the use of the so-called cell model. As was shown by Kirsh and Fuchs (1967) and by Yeh (1972), the Kuwabara flow field provides a better representation of the flow around filter fibers than the Happel flow field and has therefore been used more widely in filtration analyses (Fuchs and Stechkina, 1963; Davies, 1965; Pich, 1965; Stechkina and Fuchs, 1966; Stechkina et al., 1969; Yeh and Liu, 1974a). In the region of maximum penetrating particle size, the predominant filtration mechanisms are diffusion and interception (Stechkina et al., 1970). Consequently, only these mechanisms will be considered here, although in a somewhat different analytical manner than that used by Fuchs and Stechkina.

The approach taken here is a boundary layer approach similar to that commonly used in heat and mass transfer analysis and similar to that used by Friedlander (1957, 1958) and by Natanson (1957). In the earlier analyses by Friedlander and by Natanson, the flow fields of Tomotika and Aoi (1950) and Lamb (1932), respectively, were used. In the present case the Kuwabara flow field is used, which is believed to provide a better representation of the actual flow field in a filter than either the Tomotika–Aoi or Lamb flow fields.

DIFFUSION

Consider first the case of convective diffusion of particles of vanishingly small size, i.e., for the case of zero interception parameter, $R=R_p/R_f$ where R_p and R_f are the radii of the particle and filter fibers, respectively. In polar coordinates r

and θ the equation of convective diffusion can be written as follows:

$$u_r \frac{\partial n}{\partial r} + \frac{u_\theta}{r}\frac{\partial n}{\partial \theta} = D\left(\frac{\partial^2 n}{\partial r^2} + \frac{1}{r}\frac{\partial n}{\partial r} + \frac{\partial^2 n}{r^2 \partial \theta^2}\right), \tag{1}$$

where n is the particle concentraton, D is the diffusion coefficient, and u_r and u_θ are the radial and circumferential component flow velocities, respectively. The appropriate boundary conditions are

$$n=0 \quad \text{at} \quad r=R_f;$$

$$\frac{\partial n}{\partial r}=0, \qquad n=n_0 \quad \text{at } r=\rho(\theta), \tag{2}$$

where n_0 is particle concentraton at the outer edge of boundary layer and ρ is the radius of the boundary layer. The velocity components u_r and u_θ are given by suitable flow fields. Equation (1) is an elliptic partial differential equation that cannot be solved analytically.

In the case of the boundary layer analysis used here, the diffusion boundary layer is thin, and the term $(D/r^2)(\partial^2 n/\partial\theta^2)$ on the right-hand side of Eq. (1) is much smaller than the other terms and may be neglected to give

$$u_r \frac{\partial n}{\partial r} + \frac{u_\theta}{r}\frac{\partial n}{\partial \theta} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right). \tag{3}$$

This is equivalent to neglecting the circumferential diffusion of the particles and considering only the diffusion in the radial direction. This is a reasonable assumption, since the concentration gradient of particles in the circumferential direction is expected to be small. In order to obtain an exact analytical solution, Natanson (1957) further dropped the term $(D/r)(\partial n/\partial r)$ on the right-hand side of Eq. (1). In this study, this term is retained.

By definition of the stream function ψ the velocity components appearing in Eq. (3) can be written

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{\partial\psi}{\partial r}. \tag{4}$$

To simplify Eq. (3) further the coordinate system (r, θ) will be changed into the system $[\psi(r,\theta), \theta]$.

Utilizing the chain rule for coordinate transformation,

$$\left.\frac{\partial n}{\partial r}\right|_\theta = \frac{\partial n}{\partial\psi}\frac{\partial\psi}{\partial r} = u_\theta\frac{\partial n}{\partial\psi},$$

$$\left.\frac{\partial n}{\partial\theta}\right|_\psi = \frac{\partial n}{\partial\psi}\frac{\partial\psi}{\partial\theta} + \left.\frac{\partial n}{\partial\theta}\right|_\psi$$

$$= r u_r \frac{\partial n}{\partial\psi} + \left.\frac{\partial n}{\partial\theta}\right|_\psi. \tag{5}$$

Equation (3) can be reduced to

$$\left.\frac{\partial n}{\partial\theta}\right|_\psi = D\frac{\partial}{\partial\psi}\left(r\frac{\partial n}{\partial r}\right). \tag{6}$$

The equation has now been simplified by the change of coordinates and can be readily integrated from the fiber surface to the outer edge of the concentration boundary layer,

$$\int_0^{\psi_{\rho(\theta)}} \left.\frac{\partial n}{\partial\theta}\right|_\psi d\psi \doteq DR_f\left(\frac{\partial n}{\partial r}\right)_{r=R_f} \tag{7}$$

where $\psi_{\rho(\theta)}$ is the stream function passing through the outer diffusion boundary layer at θ. In obtaining Eq. (7), we have assumed $\partial n/\partial r = 0$ at $r=\rho$. Since

$$\frac{d}{d\theta}\int_0^{\psi_{\rho(\theta)}} n(\psi,\theta)\,d\psi$$

$$= \int_0^{\psi_{\rho(\theta)}} \frac{\partial n}{\partial\theta}\,d\psi + n_0\frac{\partial\psi_\rho}{d\theta}. \tag{8}$$

Equation (7) can be written

$$\frac{d}{d\theta}\int_0^{\psi_{\rho(\theta)}} n(\psi,\theta)\,d\psi - n_0\frac{d\psi_{\rho(\theta)}}{d\theta}$$

$$= DR_f\left(\frac{\partial n}{\partial r}\right)_{r=R_f}. \tag{9}$$

On the left-hand side of Eq. (9) are the terms representing the rate of convection of particles into the boundary layer, and on the right-hand side is the term representing the rate of diffusion of particles to the cylinder surface. Across the top of the diffusion boundary layer there is no diffusion since the concentration gradient is zero there. Equation (9) can also be obtained directly by constructing a control volume with the width and height $r\,d\theta$ and $\rho - R_f$, respectively. By applying Eq. (9) to the rule of integration by parts, we have

$$\frac{d}{d\theta}\int_0^n \psi\, dn = -DR_f\left(\frac{\partial n}{\partial r}\right)_{r=R_f}. \tag{10}$$

Using the dimensionless quantitites

$$\rho' = \frac{\rho}{R_f}, \qquad \psi' = \frac{\psi}{uR_f},$$

$$n' = \frac{n}{n_0}, \qquad r' = \frac{r}{R_f},$$

where u is the average air velocity inside the filter, Eq. (10) can be rewritten

$$\frac{d}{d\theta}\int_0^1 \psi'\, dn' = -\frac{D}{uR_f}\left(\frac{\partial n'}{\partial r'}\right)_{r'=1}, \tag{11}$$

which in turn becomes

$$\frac{dM'}{d\theta} = -\frac{2}{\mathrm{Pe}}\left(\frac{\partial n'}{\partial r'}\right)_{r'=1} \tag{12}$$

where $M'(\psi', \theta')$ is the dimensionless rate of particle diffusion to the fiber surface from $\theta=0$ to $\theta=\theta$, and $\mathrm{Pe}=u(2R_f)/D$ is the Peclet number. Thus, M' is zero at $\theta=0$ and reaches a maximum at $\theta=\pi$. In terms of the dimensionless quantities, the boundary conditions in Eq. (2) become

$$\begin{aligned} &n'=0 && \text{at} \quad r'=1, \\ &n'=1, \quad \frac{\partial n'}{\partial r'}=0 && \text{at} \quad r'=\rho'(\theta). \end{aligned} \tag{13}$$

Further, from Eq. (3) and from the definition of the boundary layer, we obtain the additional conditions,

$$\frac{\partial}{\partial r'}\left(r'\frac{\partial n'}{\partial r'}\right)=0$$

$$\text{at} \quad r'=1 \quad \text{and} \quad r'=\rho'. \tag{14}$$

It is a usual practice in boundary layer analysis to seek a dimensionless particle concentration $n'(r', \theta')$ that satisfies as many boundary conditions as may be needed. The following simple expression is used by Friedlander (1957):

$$n' = \frac{\ln r'}{\ln \rho'(\theta)}. \tag{15}$$

The stream function of the Kuwabara flow has been approximated for the present study as follows:

$$\psi' = \frac{1-\alpha}{Kr'}(r'-1)^2 \sin\theta, \tag{16}$$

where α is the solidity, packing density, or volume fraction of filter. The detailed information concerning this approximation, including the accuracy that can be obtained, is described in the Appendix.

Substituting Eqs. (15) and (16) into Eq. (12), M' can be written

$$\begin{aligned} M' &= \frac{1}{\ln\rho'}\int_1^{\rho'} \frac{1-\alpha}{K}\frac{(r'-1)^2}{r'^2}\sin\theta\, dr' \\ &= \frac{1}{\ln\rho'}\frac{1-\alpha}{K}\left(r' - 2\ln r' - \frac{1}{r'}\right)_1^{\rho}\sin\theta, \end{aligned} \tag{17}$$

where K is the Kuwabara hydrodynamic factor as defined by Eq. (A.1). The logarithms appearing in Eq. (17) can be further approximated by the following series expansion:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots,$$

$$x > \tfrac{1}{2}. \tag{18}$$

Substituting Eq. (18) into Eq. (17),

$$M' = \frac{1-\alpha}{K}\frac{(\rho'-1)^2}{3\rho'^2}\sin\theta,$$

or

$$\frac{\rho'-1}{\rho'} = \left[\frac{3K}{(1-\alpha)\sin\theta}M'\right]^{1/2}. \tag{19}$$

With this relation, Eq. (12) can now be integrated once more:

$$\left(\frac{\rho'-1}{\rho'}\right)\frac{dM'}{d\theta} = \frac{-2}{\mathrm{Pe}}, \tag{20}$$

or

$$\left(\frac{3K}{1-\alpha}\right)^{1/2}\int_0^{M_\alpha'} M'^{1/2}\, dM' = \int_0^{\pi} -\frac{2}{\mathrm{Pe}}\sin^{1/2}\theta\, d\theta, \tag{21}$$

where the limits of the integration have been given as M_α' when $\theta=\pi$ and 0 when $\theta=0$. The total number of particles diffused to the surface is obtained as

$$M_\alpha' = \left[\frac{3}{2}\left(\frac{1-\alpha}{3K}\right)^{1/2}\frac{2}{\mathrm{Pe}}\sqrt{\pi}\,\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right]^{2/3}$$

$$=2.6\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{-2/3}, \quad (22)$$

where $\Gamma(c)$ is the gamma function of an arbitrary constant c. By definition of the single fiber efficiency due to diffusion η_D and by Eq. (12), we have

$$\eta_D = DR_f\int_0^\pi\left(\frac{\partial n}{\partial r}\right)_{r=R_f} d\theta/uR_f n_0$$

$$=\frac{2}{\mathrm{Pe}}\int_0^\pi\left(\frac{\partial n'}{\partial r'}\right)_{r'=1} d\theta$$

$$=M_\alpha' = 2.6\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{-2/3}. \quad (23)$$

The result shows that for pure diffusion, the single fiber efficiency is proportional to $[(1-\alpha)/K]^{1/3}\mathrm{Pe}^{-2/3}$.

In Table 1 the theoretical results obtained by various investigators on the diffusive collection of particles by fibrous filters are listed. It can be seen that all predict that $\eta_D \propto \mathrm{Pe}^{-2/3}$. It should be noted that for diffusion in ideal frictionless flow, $\eta_D \propto \mathrm{Pe}^{-1/2}$.

Concerning the dependence of η_D on filter solidity, we note that because the flow fields used by Friedlander (1957), Natanson (1957), and Stechkina (1966) are those of Lamb or Tomotika and Aoi, their results do not contain provisions for varying filter solidity. Later Stechkina and Fuchs (1966) obtained their expression (listed in Table 1) by neglecting all the terms in Eq. (A.1) that included the solidity α. Under most circumstances this seems to be a reasonable simplification, since α is usually quite small. However, when the solidity is increased or when the diffusion layer thickness becomes large, namely, when Pe becomes very small, the terms are not completely negligible. By approximating these terms as described in the Appendix and through a somewhat different approach, we have obtained an expression that can account for the effect of filter solidity to a first approximation. It is believed that with the inclusion of the factor $1-\alpha$ in the theoretical expresson, the results can be applied over a wider range of conditions than the expressions without the factor. Further discussion and comparison of the theoretical expression with the experiment will be made later.

TABLE 1. List of Theoretical Diffusional Filtration Efficiencies for Fiber Normal to Flow Direction

Investigators	Theoretical prediction	Flow field used	Remarks
Langmuir (1942)	$1.70\,K_0^{-1/3}\,\mathrm{Pe}^{-2/3}$	Lamb's flow	$K_0 = 2 - \ln \mathrm{Re}$
Natanson (1957)	$2.9\,K_0^{-1/3}\,\mathrm{Pe}^{-2/3}$	Lamb's flow	$K_0 = 2 - \ln \mathrm{Re}$
Friedlander (1957)	$3.25\,A^{1/3}\,\mathrm{Pe}^{-2/3}$	Tomotika and Aoi	A is a constant depending on Re and α
Stechkina (1966)	$2.9\,K_0^{-1/3}\,\mathrm{Pe}^{-2/3} + 0.62\,\mathrm{Pe}^{-1}$	Lamb's flow	$K_0 = 2 - \ln \mathrm{Re}$
Stechkina and Fuchs (1966)	$2.9\,K^{-1/3}\,\mathrm{Pe}^{-2/3} + 0.62\,\mathrm{Pe}^{-1}$	Kuwabara flow	$K = -\frac{1}{2}\ln\alpha - 0.75 + \alpha - \frac{1}{4}\alpha^2$
Spielman and Goren (1968)	$2.9\,K_s^{-1/3}\,\mathrm{Pe}^{-2/3}$	Brinkman flow	$K_s = k_0(\lambda)/\lambda k_1(\lambda)$, where $\lambda = 4\alpha(\lambda/2 + 1/K_s)$ and k_0, k_1 are modified Bessel functions
present study	$2.6\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{-2/3}$	Kuwabara flow	$K = -\frac{1}{2}\ln\alpha - 0.75 + \alpha - \frac{1}{4}\alpha^2$

INTERCEPTION

Direct interception takes place on account of the finite size of the particles. The single fiber efficiency due to interception η_R is defined

$$\eta_R = Y/R_f = uY/uR_f, \tag{24}$$

where Y is the distance between the center line and the streamline below which all the particles are collected. Since an approaching streamline at a distance Y from the stagnation streamline must later pass through a point at a distance R_p from the fiber surface at $\theta = \pi/2$ in order for the particle to be collected, we have

$$\eta_R = -\frac{\psi_{R_f + R_p, \pi/2}}{uR_f} = \psi'_{1+R, \pi/2}, \tag{25}$$

where $\psi'_{1+R,\pi/2}$ is the dimensionless stream function passing through a point at a distance of R_p from the fiber surface at an angle $\pi/2$. Substituting the stream function expression for the Kuwabara flow in Eq. (A.1), we have

$$\eta_R = \frac{1+R}{2K}\left[2\ \ln\ (1+R) - 1 + \alpha + \left(\frac{1}{1+R}\right)^2\left(1 - \frac{\alpha}{2}\right) - \frac{\alpha}{2}(1+R)^2\right]. \tag{26}$$

Although Eq. (26) is a complete expression for the interception efficiency based on Kuwabara flow field, the form of the equation is somewhat long, and it would be useful to reduce it to a simpler form. Since we have already obtained an approximation form of the stream function as given in the Appendix, we simply substitute Eq. (A.9) into Eq. (25) with $1+R$ for r' and $\pi/2$ for θ:

$$\eta_R = \frac{1-\alpha}{K}\frac{R^2}{1+R}. \tag{27}$$

It may be seen that the filter efficiency expression can thus be greatly simplified. As discussed in the Appendix, the primary limitation of the approximation is that $r/R_f \sim 1$, and this limitation corresponds to $1+R \sim 1$, or that R is small. Again, the condition of solidity being small is not too restrictive, as is discussed in the Appendix.

In the following discussion we compare the present approximation with the approximations obtained by other investigators. Stechkina and Fuchs (1966) approximated Eq. (26) by omitting all the terms containing α and obtained the equation

$$\eta_R = \frac{1+R}{2K}\left[2\ \ln\ (1+R) - 1 + \frac{1}{(1+R)^2}\right]. \tag{28}$$

The limitation of their approximation is that both the solidity α and R must be small. Natanson (1957, 1962) approximated the interception efficiency:

$$\eta_R = (1/K_0)R^2, \tag{29}$$

where K_0 was originally obtained from Lamb's flow, although it can be replaced by K in the case of Kuwabara flow field.

It is of interest to note that if R becomes extremely small, the present approximation, Eq. (27), reduces to

$$\eta_R = \frac{1-\alpha}{K}R^2 \qquad \text{for} \quad R \to 0. \tag{30}$$

This can also be derived if Eq. (A.9) is further approximated as

$$\psi' = \frac{1-\alpha}{K}(r'-1)^2 \sin\theta, \qquad r' \to 1.$$

However, the magnitude of the error involved in this approximation becomes in general larger than that given by Eq. (A.11). If α is extremely small, Eq. (27) reduces to

$$\eta_R = \frac{1}{K}\frac{R^2}{1+R} \qquad \text{for} \quad \alpha \to 0. \tag{31}$$

In case both R and α are small, Eq. (27) reduces to the result of Natanson (1962), given by Eq. (29). Thus, it is apparent that Eq. (27) would be less restrictive with regard to both R and α.

Table 2 is a comparison of values for the interception efficiency for a few selected cases as calculated with the preceding equations. It can be seen that when both R and α are small, all five approximations give efficiency values that are close to the value computed with the full Kuwabara form, Eq. (26). However, when α

TABLE 2. Comparison of the Interception Efficiencies Computed with Different Approximations

Solidity α	Diameter ratio R	Hydrodynamic factor K	Efficiency using the original flow, Eq. (26)	Present Study Eq. (27)	Present Study Eq. (30)	Present Study Eq. (31)	Stechkina and Fuchs (1966), Eq. (23)	Natanson (1962), Eq. (29)
0.001	0.05	2.7049	0.00089	0.00088	0.00092	0.00088	0.00089	0.00092
0.001	0.1	2.7049	0.00347	0.00336	0.00370	0.00336	0.00347	0.00370
0.005	0.05	1.9042	0.00126	0.00124	0.00131	0.00125	0.00127	0.00131
0.01	0.1	1.5626	0.00594	0.00576	0.00634	0.00582	0.00601	0.00640
0.1	0.1	0.4988	0.0168	0.0164	0.0180	0.0182	0.0188	0.0200
0.1	0.2	0.4988	0.0630	0.0602	0.0733	0.0668	0.0711	0.0802
0.2	0.2	0.2447	0.112	0.109	0.1308	0.1360	0.145	0.163
0.3333	0.01	0.1049	0.00063	0.00063	0.00064	0.00094	0.00095	0.00095
0.3333	0.1	0.1049	0.0577	0.0578	0.0636	0.0867	0.0895	0.0954
0.5	0.01	0.0341	0.00145	0.00145	0.00146	0.00290	0.00291	0.00293
0.5	0.1	0.0341	0.1283	0.1333	0.1466	0.2665	0.2753	0.2933

becomes large, the present approximation, Eq. (27), gives much closer values than that of Stechkina and Fuchs [Eq. (28)] in spite of its simplified form.

COMBINED DIFFUSION AND INTERCEPTION

One of the simplest ways to combine the diffusion and the interception mechanisms with a reasonable accuracy is to add the two individual efficiencies to obtain the combined efficiency. This practice is based on the assumption that only one mechanism is predominant, the contribution made by the other mechanism being small. This assumption has been found to be adequate for combining the diffusion and the interception mechanisms. As shown in Eq. (23), the efficiency due to diffusion decreases with increasing particle size, whereas the efficiency due to interception increases rapidly with increasing particle size [see Eq. (27)]. Therefore, this simple practice has given reasonable results for the combined efficiencies.

With the above assumption, and assuming that impaction is unimportant, we have

$$\eta=\eta_D+\eta_R; \tag{32}$$

by means of Eqs. (23) and (27), we have

$$\eta=2.6\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{-2/3}+\left(\frac{1-\alpha}{K}\right)\frac{R^2}{1+R}. \tag{33}$$

Equation (33) is the final equation for the single fiber efficiency for diffusion and interception for the case where the filter fibers are transverse to the flow. This result is plotted in Figure 1, together with the results of Yeh and Liu (1974a) and Stechkina et al. (1969). Yeh and Liu considered diffusion, interception, and inertial impaction simultaneously and solved for filtration efficiency numerically, while Stechkina et al. used a separate expression for each filtration mechanism plus a combination term for diffusion and interception to fit their numerical results. It is seen in Figure 1 that Eq. (33) is in satisfactory agreement with these theoretical results for the region where $\mathrm{Pe}\lesssim 30{,}000$ and $R\lesssim 0.2$. This comparison further confirms that the effect of inertial impaction mechanism in the indicated filtration regime is indeed not significant and that the two asymptotic expressions for diffusional and interceptional filtration efficiencies can be additive.

Although Eq. (33) can be directly used for practical applications, it is more convenient for

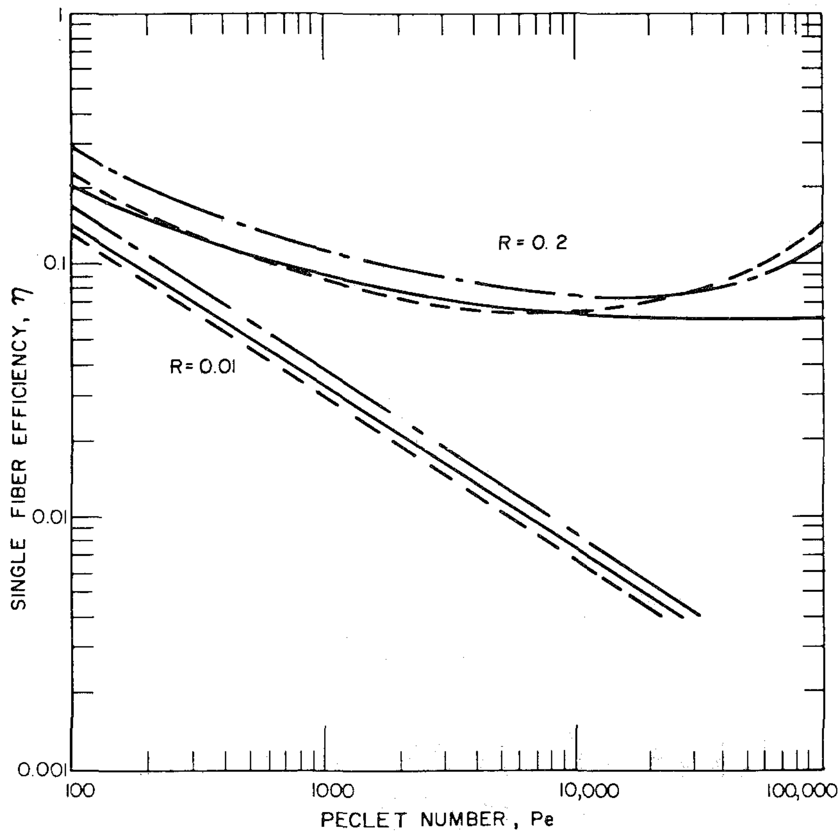


FIGURE 1. Comparison of Eq. (33) (———) with the theories of Stechkina et al. (1969) (– – –) and Yeh and Liu (1974a) (— · —).

purposes of correlating experimental data to modify the equations as follows. Following the method used by Friedlander (1958) and Spielman and Goren (1969), we multiply both sides of Eq. (33) by $\mathrm{Pe}R/\sqrt{1+R}$. We then obtain

$$\eta \mathrm{Pe}\frac{1}{\sqrt{1+R}} = 2.6\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{1/3}\frac{R}{\sqrt{1+R}} + \left(\frac{1-\alpha}{K}\right)\mathrm{Pe}\frac{R^3}{(1+R)^{3/2}}$$
$$= 2.6\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{1/3}\frac{R}{\sqrt{1+R}} + \left[\left(\frac{1-\alpha}{K}\right)^{1/3}\mathrm{Pe}^{1/3}\frac{R}{\sqrt{1+R}}\right]^3. \tag{34}$$

This equation states that in the diffusion and interception regime the quantity $\eta \mathrm{Pe}R/\sqrt{1+R}$ should be a single-valued function of the parameter $[(1-\alpha)/K]^{1/3}\mathrm{Pe}^{1/3}R/\sqrt{1+R}$. Figure 2 is the plot of Eq. (34) using these parameters. The theory of Yeh and Liu for $\alpha=0.1$, $D_f=10\ \mu$m, and $U_0=1$ cm/sec is included for comparison.

EMPIRICAL CORRELATION

Most real filters are made of fibers that are randomly oriented, rather than perpendicular to the plane as adopted in the present model. Often times, fibers are not uniformly distributed and phenomena such as channel flow and shadowing can occur. The existing theories take into account the nonideal characteristics of the

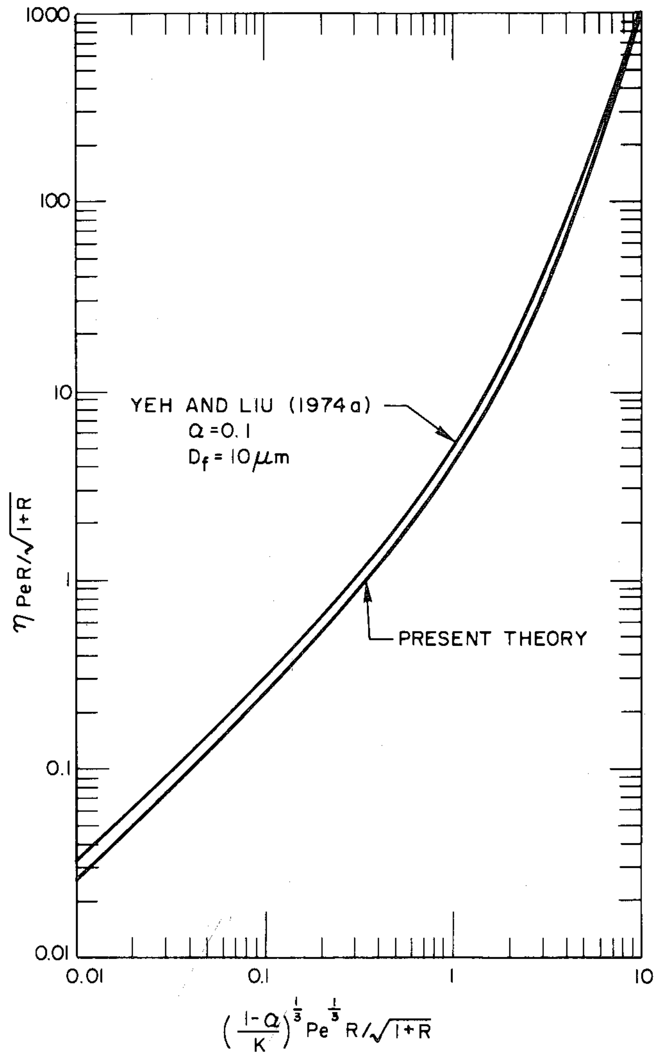


FIGURE 2. Comparison of Eq. (34) with the theory of Yeh and Liu (1974a).

fibrous filters by defining an empirical factor such as effective fiber diameter factor (Davies, 1952), inhomogeneity factor (Stechkina et al., 1969), or effective fiber length (Yeh and Liu, 1974b). This factor is obtained by comparing the experimental pressure drop and that predicted by the flow field. Although this method is found to be useful, it is difficult to justify such a direct incorporation of pressure drop into the mechanisms giving filtration efficiencies. In the present study, therefore, the relationships between the parameters $\eta \mathrm{Pe} R/\sqrt{1+R}$ and $[(1-\alpha)/K]^{1/3} \times \mathrm{Pe}^{1/3} R/\sqrt{1+R}$ will be determined empirically using experimental data (Lee, 1977; Lee and Liu, 1982); the constants appearing on the right-hand side of Eq. (33) or (34) will then be deduced from the experimental data.

For the purpose of comparing with experimental data, we first write Eq. (33) as follows:

$$\eta = \beta_1 \left(\frac{1-\alpha}{K}\right)^{1/3} \mathrm{Pe}^{-2/3} + \beta_2 \left(\frac{1-\alpha}{K}\right) \frac{R^2}{1+R}, \tag{35}$$

where β_1 and β_2 are empirical constants to be derived from the experimental data. With these correlation constants, Eq. (34) then becomes

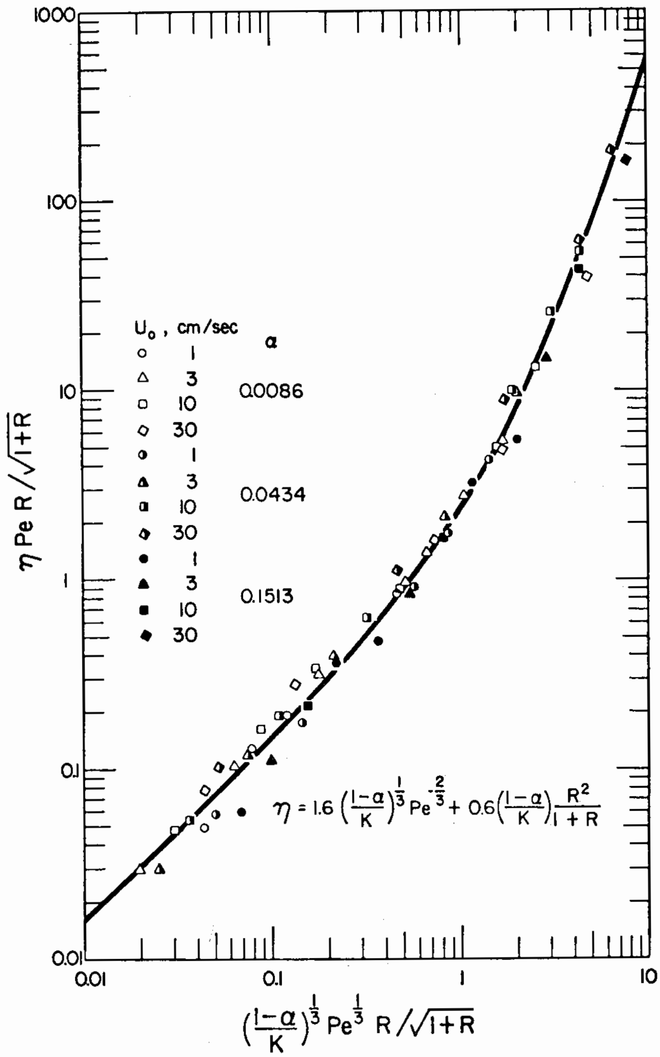


FIGURE 3. Correlation of filtration data in the form of Eq. (35).

$$\eta \mathrm{Pe} \frac{R}{\sqrt{1+R}} = \beta_1 \left(\frac{1-\alpha}{K}\right)^{1/3} \mathrm{Pe}^{1/3} \frac{R}{\sqrt{1+R}} + \beta_2 \left[\left(\frac{1-\alpha}{K}\right)^{1/3} \mathrm{Pe}^{1/3} \frac{R}{\sqrt{1+R}}\right]^3. \quad (36)$$

In Figure 3 the quantity $\eta \mathrm{Pe} R/\sqrt{1+R}$ is plotted against $[(1-\alpha)/K]^{1/3}\mathrm{Pe}^{1/3}R/\sqrt{1+R}$ using the experimental data for Dacron filters with solidities of 0.0086, 0.0474, and 0.151. The particle sizes ranged from 0.05 to 1.3 μm. The corresponding interception parameter was from 0.0045 to 0.12. The maximum Stokes number was 0.22. Figure 3 shows that nearly all the experimental data fall on a single-valued curve, as the theory has predicted. Further, Eq. (36) shows that the slope of the curve on this log–log plot should be 1 for small values of the parameter $[(1-\alpha)/K]^{1/3}\mathrm{Pe}^{1/3}R/\sqrt{1+R}$ and approaches 3 for large values. These are indeed realized, indicating that the analysis is valid and that the simple procedure of adding the single-fiber efficiencies due to diffusion and interception is indeed correct. Further, the constants β_1 and β_2 are found to have the values 1.6 and 0.6, respectively. This leads to the following equation for the empirical correlation:

$$\eta \mathrm{Pe} \frac{R}{\sqrt{1+R}} = 1.6 \left(\frac{1-\alpha}{K}\right)^{1/3} \mathrm{Pe}^{1/3} \frac{R}{\sqrt{1+R}} + 0.6 \left(\frac{1-\alpha}{K}\right) \mathrm{Pe} \frac{R^3}{(1+R)^{3/2}}, \qquad (37)$$

and the equation

$$\eta = 1.6 \left(\frac{1-\alpha}{K}\right)^{1/3} \mathrm{Pe}^{-2/3} + 0.6 \left(\frac{1-\alpha}{K}\right) \frac{R^2}{1+R} \qquad (38)$$

for the single-fiber efficiency.

It can be easily seen from Figure 3 that the value of η is due almost entirely to the contribution of the first term in Eq. (38) when $[(1-\alpha)/K]^{1/3}\mathrm{Pe}^{1/3}R/\sqrt{1+R} < 0.3$, and the second term predominates when the value of this parameter is larger than 3. This result can be used as a criterion for determining whether filtration can be considered as pure diffusion or as pure interception. When the value of the parameter is smaller than 0.3, the diffusion mechanism predominates. When the value is larger than 3, interception predominates. However, in the range from 0.3 to 3, both mechanisms are important.

Other forms of empirical correlations can be suggested by the different theoretical equations for the diffusion and interception efficiencies. For instance, using Eqs. (23) and (30) for the diffusion and interception efficiencies, we should expect an empirical correlation between the parameters $\eta \mathrm{Pe}R$ and $[(1-\alpha)/K]^{1/3}\mathrm{Pe}^{1/3}R$. Similarly, using the Stechkina–Fuchs equation for the diffusion efficiency (but dropping the second term in the equation shown in Table 1) and Natansons's equation [Eq. (29)] for interception, we should expect an empirical correlation between the parameters $\eta \mathrm{Pe}R$ and $K^{-1/3}\mathrm{Pe}^{1/3}R$. The degree of correlation between these parameters was found to be about equal to the one shown in Figure 3. However, it is believed that with a wider range of values in α or R the present correlation should improve over these. Similarly, the original analysis of Friedlander (1958) suggested a correlation between the parameters $\eta \mathrm{Pe}R$ and $\mathrm{Pe}^{1/3}\mathrm{Re}^{1/3}R$. This correlation, shown in Figure 4, gives somewhat greater scatter, especially for high values of $\mathrm{Pe}^{1/3}\mathrm{Re}^{1/3}R$, and consequently is less satisfactory.

To observe how the semiempirical correlation of Eq. (38) can reproduce the experimental data, the comparisons shown in Figures 5 and 6 have been prepared. It is seen that the variation of efficiency with respect to size and the variation with respect to solidity are both predicted with satisfactory accuracy by means of Eq. (38) at different filtration velocities.

CONCLUSIONS

The filtration theory for diffusion and interception developed in the study has a rather simple form, yet it has been found successful in correlating filtration efficiency in the region of maximum penetrating particle size with good accuracy. It was also demonstrated that the present approximation can better represent the filter solidity dependence of the filtration efficiency than the other comparable theories.

The successful comparison of the present theory with the experimental data indicates that the main filtration mechanisms involved in the region of minimum efficiency are diffusion and interception, with the impaction mechanism playing only a minor role.

APPENDIX

THE KUWABARA FLOW FIELD

The Kuwabara flow field represents a solution of the two-dimensional viscous flow equation for a system of cylinders placed transverse to the flow. By drawing an imaginary coaxial boundary around a cylinder, the filter solidity α can be

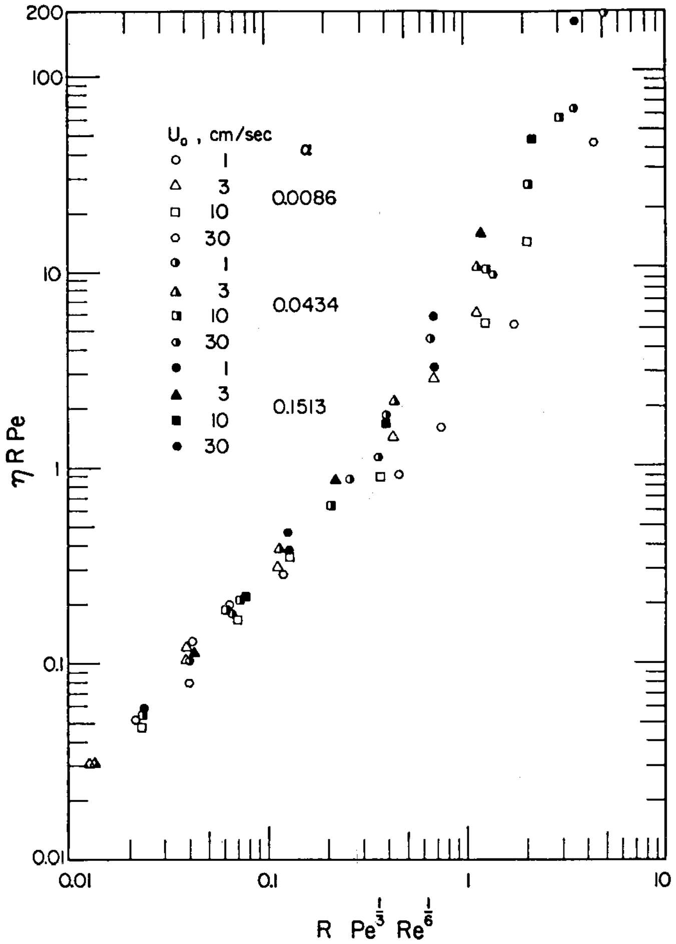


FIGURE 4. Correlation of filtration data in the form suggested by Friedlander's analysis.

made equal to that of the coaxial cylinders, i.e., $\alpha = R_f^2/b^2$. Based on the assumption of vanishing vorticity on the outer boundary, Kuwabara obtained the following expressions for the stream function and velocity components:

$$\psi = \frac{ur}{2K}\left[2\ln\frac{r}{R_f} - 1 + \alpha + \frac{R_f^2}{r^2}\left(1-\frac{\alpha}{2}\right) - \frac{\alpha}{2}\frac{r^2}{R_f^2}\right]\sin\theta,$$

$$u_r = \frac{u}{2K}\left[2\ln\frac{r}{R_f} - 1 + \alpha + \frac{R_f^2}{r^2}\left(1-\frac{\alpha}{2}\right) - \frac{\alpha}{2}\frac{r^2}{R_f^2}\right]\cos\theta,$$

$$u_\theta = \frac{u}{2K}\left[2\ln\frac{r}{R_f} + 1 + \alpha - \frac{R_f^2}{r^2}\left(1-\frac{\alpha}{2}\right) - \frac{3\alpha}{2}\frac{r^2}{R_f^2}\right]\sin\theta, \tag{A.1}$$

where $K = -\frac{1}{2}\ln\alpha - \frac{3}{4} + \alpha - \frac{1}{4}\alpha^2$ is called the Kuwabara hydrodynamic factor.

Although the expression for the stream function in the original Kuwabara flow field does not appear to be very complicated, it is necessary to reduce it to a simpler form for certain applications in theoretical analyses. To obtain an approximation under certain conditions, let us introduce the dimensionless quantities

$$r' = r/R_f, \qquad \psi' = \psi/uR_f, \tag{A.2}$$

and write the stream function appearing in Eq.

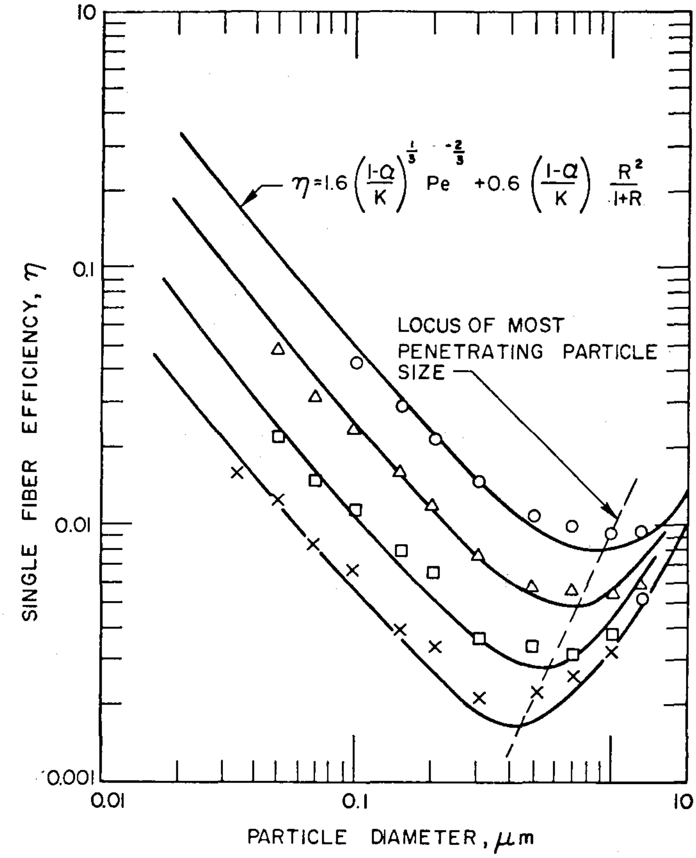
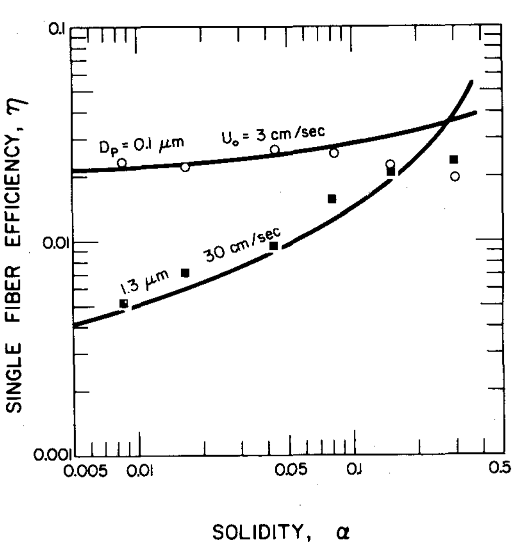


FIGURE 5. Comparison of data with correlation equation for Dacron filter ($\alpha = 0.0086$).

FIGURE 6. Single-fiber efficiency as a function of filter solidity—comparison of theory and experiment.



(A.1) in dimensionless form:

$$\psi' = (r'/2K)[(2\ln r' - 1 + 1/r'^2) - \alpha(-1 + 1/2r'^2 + \tfrac{1}{2}r'^2)] \sin\theta. \tag{A.3}$$

Using the series expansion for $\ln r'$ valid for $r' > \frac{1}{2}$, we have

$$\ln r' = \frac{r'-1}{r'} + \frac{1}{2}\left(\frac{r'-1}{r'}\right)^2 + \frac{1}{3}\left(\frac{r'-1}{r'}\right) + \cdots. \tag{A.4}$$

Using the first two terms as an approximation, we have

$$\psi' \simeq \frac{r'}{2K}\left[\frac{2(r'-1)^2}{r^2} - \alpha\frac{(r'^2-1)^2}{2r'^2}\right] \sin\theta. \tag{A.5}$$

It should be noted that r' has been assumed to be close to unity so that the approximation is good only for the region near the fiber surface. The error in the approximation is

$$\Delta\psi_1' = \frac{-r'}{K}\sum_{n=3}^{\infty}\frac{1}{n}\left(\frac{r'-1}{r'}\right)^n \sin\theta. \quad (A.6)$$

To make the expression even simpler, we further approximate Eq. (A.5) as

$$\psi' \simeq \frac{r'}{K}\left[\frac{(r'-1)^2}{r'^2} - \alpha\frac{(r-1)^2}{r'^2}\right]\sin\theta, \quad (A.7)$$

under the assumptions $r' \sim 1$ and $\alpha \sim 0$. The magnitude of the error involved in the second approximation is

$$\Delta\psi_2' = \frac{\alpha}{4K}\frac{(r'+3)(r'-1)^3}{r'}\sin\theta. \quad (A8)$$

It can be noted that the errors in the two approximations tend to cancel each other, since $\Delta\psi_2'$ is positive. Thus, the stream function can be written

$$\psi' = [(1-\alpha)/Kr'](r'-1)^2 \sin\theta, \quad (A.9)$$

or, in dimensional terms,

$$\psi = [u(1-\alpha)/Kr](r-R_f)^2 \sin\theta. \quad (A.10)$$

The error involved in the use of the approximate Eq. (A.9) is

$$\Delta\psi' = \Delta\psi_1' + \Delta\psi_2'$$

$$= \frac{r'}{K}\left[-\sum_{n=3}^{\infty}\frac{1}{n}\left(\frac{r'-1}{r'}\right)^n + \frac{\alpha}{4}\frac{(r'+3)(r'-1)^3}{r'^2}\right]\sin\theta. \quad (A.11)$$

Equation (A.11) indicates that the magnitude of error involved in the present approximation is $\Delta(r'-1)^3$, and this should be very small for $r' \sim 1$. In order to check this further, ψ' has been calculated using Eq. (A.3) for the case where $\alpha = 0.01$, $r' = 1.1$, and $\theta = \pi/2$ and found to be 0.00594. On the other hand, Eq. (A.9) gives the value 0.00576.

In order to determine the limitations due to the use of Eq. (A.9) in place of Eq. (A.3), let us examine Eq. (A.11) in further detail. As mentioned earlier, the conditions that $r' \sim 1$ and $\alpha \sim 0$ are the principal limitations. However, these limitations can be somewhat eased under the following circumstances. In most applications of fibrous filter α is small, and therefore $r' \sim 1$ is the only limitation. In such cases, $\Delta\psi_1$ is larger than $\Delta\psi_2$ in Eq. (A.9), resulting in an underestimation of the value of the stream function. If r' is maintained very close to unity, Eq. (A.11) can be further written

$$\lim_{r' \to 1} \Delta\psi' \simeq (1/K)(-\tfrac{1}{3}+\alpha)(r'-1)^3 \sin\theta \quad (A.12)$$

The equation indicates that the error involved in the present approximation approaches zero with α approaching $\frac{1}{3}$. This means that the approximation becomes better when α is close to $\frac{1}{3}$ as long as $r' \sim 1$. For example, with $\alpha = \frac{1}{3}$, $r' = 1.1$, and $\theta = \pi/2$, Eqs. (A.3) and (A.9) give $\psi' = 0.05765$ and 0.05779, respectively. It should be noted that the error is smaller than that shown in the preceding example in spite of the increasing solidity. It is apparent that Eq. (A.9) should remain a good approximation for α well over $\frac{1}{3}$, say 0.4 or 0.5. It should also be noted that for such high solidities, $\Delta\psi_2'$ is larger than $\Delta\psi_1'$ in absolute magnitude, and Eq. (A.9) gives a higher value for ψ'.

The velocity components can be obtained from the stream function [Eq. (4)] in the following way:

$$u_\theta' = \frac{-\partial\psi'}{\partial r} = \frac{2(1-\alpha)}{Kr'}(r'-1)\sin\theta - \frac{(1-\alpha)}{K}(r'-1)^2\frac{1}{r'^2}\sin\theta$$

$$\simeq \frac{2(1-\alpha)}{Kr'}(r'-1)\sin\theta, \quad (A.13)$$

or, in dimensional form,

$$u_\theta = \frac{2(1-\alpha)u}{K}\left(\frac{r-R_f}{r}\right)\sin\theta. \quad (A.14)$$

Similarly,

$$u_r' = \frac{1}{r'}\frac{\partial \psi'}{\partial \theta}$$

$$\doteq \frac{1-\alpha}{Kr'^2}(r'-1)^2 \cos\theta \quad \text{(A.15)}$$

or

$$u_r = \frac{(1-\alpha)u}{K}\left(\frac{r-R_f}{r}\right)^2 \cos\theta. \quad \text{(A.16)}$$

Equations (A.14) and (A.16) also can be compared with the original Kuwabara expressions appearing in Eqs. (A.1). As mentioned, Stechkina and Fuchs (1966) approximated Eqs. (A.1) by omitting all the terms containing α. Owing to the omission of α, their approximation becomes less accurate than the present approximation as α is increased.

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